Theoretical analysis of SPH in simulating free-surface viscous flows

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***Abstract*—A theoretical analysis on the performance, close to a free surface, of the most used SPH formulations for Newtonian viscous terms is carried out in this paper. After an introduction of the SPH formalism, the SPH expressions for the viscous term in the momentum equation are analyzed in their continuous form. Using a Taylor expansion, a reformulation of those expressions is undertaken which allows to characterize the behavior of the viscous term close to the free surface. Under specific flow conditions, we show that the viscous term close to the free surface is singular when the spatial resolution is increased. This problem is in essence related to the incompleteness of the kernel function close to the free surface and appears for all the formulations considered. In order to assess the impact of such singular behavior, an analysis of the global energy dissipation is carried out, which shows that such a free-surface singularity vanishes when the integral quantities are considered. Notwithstanding that, not all the SPH viscous formulas allow the correct evaluation of the energy dissipation rate and, consequently, they may lead to an inaccurate modelling of viscous free-surface flows.**

I. Introduction

Smoothed Particle Hydrodynamics (SPH) is a method devised to obtain approximate numerical solutions of the Navier-Stokes equations. The SPH discretization of these equations does not lie on a mesh-based decomposition of the fluid domain, but it relies on the transport of the fluid properties by a set of particles.

Due to its underlying structure, SPH was initially thought to solve astrophysical problems [1], [2] and a few years later, its application range was extended to gravity related incompressible inviscid free-surface flows [3]. In the first formulations, the incompressibility was achieved by choosing a stiff equation of state and an adequate numerical sound speed.

In order to extend the method to viscous flows, and due to several disadvantages that arise from the direct SPH interpolation of the second derivatives [4], some numerical viscous terms have been developed. The most widespread is surely the Monaghan & Gingold one [5]. This is main due to the fact that this formula preserves both linear and angular momenta. Another famous formulation is the one given by Morris *et al.* [6] which, although does not conserve exactly angular momentum, it is the straightforward discretization of the Laplacian using the general differentiation formulas of Español et al [7]. To recover the conservation properties, Watkins *et al.* [8], Bonet & Lok [9] and Flebbe *et al.* [10] obtained formulas that preserve both linear and angular momentum. Anyway they were extremely expensive from a computational point of view because they need to evaluate two times the particles' interactions. Due to this drawback these formulations are rarely used in practice. A further important contribution to the analysis of viscosity in SPH using direct differentiation to obtain second derivatives has been given by Takeda *et al.* [11]. Though their formula has not been used much for free surface flows, and though it does not conserve angular momentum, it allows to take into account the effects related to the compressibility of the fluid.

This paper is organized as follows: the first part (section II), is dedicated to present the flow field continuum equations, emphasizing the structure of the Newtonian stress tensor and the free surface boundary conditions for viscous flows. In section III, the SPH formalism and notation is presented. Section IV is the core of the paper, presenting an analysis of the behavior of the viscous term near the free surface using the Monaghan & Gingold formula [5] and the Morris *et al.* [6] formulation both in their continuous form. In section V, we study the global energy dissipation corresponding to those viscous formulas. Finally two test cases aimed at demonstrating the validity of the conclusions obtained from the continuous analysis are presented.

II. Governing equations

A. Field equations

The compressible Navier-Stokes equations for a barotropic fluid in Lagrangian formalism are the continuum model of the flow:

$$\begin{cases} \dfrac{D\rho}{Dt} + \rho\,\nabla\cdot\boldsymbol{u} = 0, \quad \dfrac{D\boldsymbol{u}}{Dt} = \boldsymbol{g} + \dfrac{\nabla\cdot\mathbb{T}}{\rho}, \\ p = c_0^2(\rho-\rho_0), \end{cases} \tag{II.1}$$

In these equations, ρ is the fluid density, ρ_0 is the reference density, p is the pressure, c_0 is the sound speed (assumed constant) and $\boldsymbol{g}$ is a generic external volumetric force field.

The flow velocity, $\boldsymbol{u}$, is defined as the material derivative of a fluid particle position $\boldsymbol{r}$:

$$\frac{D\boldsymbol{r}}{Dt} = \boldsymbol{u}, \tag{II.2}$$

$\mathbb{T}$ is the stress tensor of a Newtonian fluid:

$$\mathbb{T} = (-p + \lambda \operatorname{tr}\mathbb{D})\,\mathbb{1} + 2\mu\mathbb{D}, \tag{II.3}$$

with $\mathbb{D}$ being the rate of strain tensor - $\mathbb{D} = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)/2$ - and μ, λ the Lamé parameters. For the analysis which follows, it is useful to consider the viscous part of the stress tensor:

$$\mathbb{V} = \lambda \operatorname{tr}\mathbb{D}\,\mathbb{1} + 2\mu\mathbb{D}. \tag{II.4}$$

With this notation, and assuming that the Lamé parameters are constant in the fluid domain, the divergence of the viscous part of the stress tensor, which will be analyzed through this work, will look like:

$$\nabla\cdot\mathbb{V} = (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u}. \tag{II.5}$$

In order to close this model, it is necessary to specify the boundary conditions (BC).

1) Boundary conditions: The fluid domain will be denoted as Ω and the only boundary considered will be a free surface $\partial\Omega_F$. Both a kinematic and a dynamic BC are imposed at the free surface. The kinematic free-surface BC (hereinafter KFSBC) implies that the material points initially on $\partial\Omega_F$ will remain in $\partial\Omega_F$ as this region evolves with the fluid flow. This is formalized by projecting both the particle speed $\boldsymbol{u}$ and the boundary speed $V_{\partial\Omega_F}$ onto the boundary normal pointing out of the fluid $\boldsymbol{n}$, as:

$$\boldsymbol{u}\cdot\boldsymbol{n} = V_{\partial\Omega_F}\cdot\boldsymbol{n} \qquad \forall\boldsymbol{x}\in\partial\Omega_F \tag{II.6}$$

The dynamic free-surface BC (hereinafter DFSBC) is a consequence of the continuity of the stresses across the free surface. Assuming that surface tension is negligible, a free surface does not stand neither perpendicular normal stresses nor parallel/tangential shear stresses. For a Newtonian fluid, by denoting such stress field as $\boldsymbol{t}$, the DFSBC can be expressed as:

$$\boldsymbol{t} = \mathbb{T}\cdot\boldsymbol{n} = (-p + \lambda\operatorname{tr}\mathbb{D})\,\boldsymbol{n} + 2\mu\mathbb{D}\cdot\boldsymbol{n} = 0. \tag{II.7}$$

This condition can be projected both in the normal direction and in the free-surface tangent plane ($\boldsymbol{\tau}$ being a unitary vector lying on such tangent plane). Considering that $\operatorname{tr}\mathbb{D} = \nabla\cdot\boldsymbol{u}$ and $\boldsymbol{n}\cdot\mathbb{D}\cdot\boldsymbol{n} = \boldsymbol{n}\cdot\partial\boldsymbol{u}/\partial\boldsymbol{n}$, two relationships arise from the projection of equation (II.7):

$$p = \lambda\nabla\cdot\boldsymbol{u} + 2\,\mu\,\boldsymbol{n}\cdot\partial\boldsymbol{u}/\partial\boldsymbol{n} \tag{II.8}$$

$$\boldsymbol{\tau}\cdot\mathbb{D}\cdot\boldsymbol{n} = 0 \tag{II.9}$$

An important consequence of the equation (II.8) is that the pressure field is in general discontinuous across the free surface.

III. The continuous SPH model

The SPH continuous model if based on the filtering (*smoothing*) of any generic flow field f with a convolution integral over the fluid domain Ω

$$\langle f\rangle(\boldsymbol{r}) = \int_\Omega f(\boldsymbol{r}')\,W(\boldsymbol{r}'-\boldsymbol{r};h)\,dV' \tag{III.10}$$

$W(\boldsymbol{r}'-\boldsymbol{r};h)$ is a weight function, which in practical applications must have a compact support $\Omega(\boldsymbol{r})$, and h (usually referred to as the *smoothing length*) is a characteristic length of such support (see figure 1). The smoothing length h, which will be considered constant in this study, is from a physical perspective, representative of the size of the domain of influence of the fluid particle which is at the position indicated by $\boldsymbol{r}$. The weight function $W(\boldsymbol{r}'-\boldsymbol{r},h)$, called *smoothing function* or *kernel* in the SPH terminology, is positive, radial centered in $\boldsymbol{r}$ and decreases monotonously with the distance $|\boldsymbol{r}-\boldsymbol{r}'|$. This monotonous decrease gets to zero at the border of its support $\Omega(\boldsymbol{r})$. The kernel that will be considered in this study is supposed to be isotropic in space, which is equivalent to being dependent only on the distance $s = |\boldsymbol{r}'-\boldsymbol{r}|$. The notation $W(\boldsymbol{r}'-\boldsymbol{r};h)$ will be shortened hereinafter as $W(\boldsymbol{r}'-\boldsymbol{r})$ and the dependence on h will be implicitly assumed. In the limit as the smoothing length h goes to zero, the original field of the convolution integral (III.10) should be recovered. In order to be so, the kernel W must integrate to one (see e.g. [12]).

$$\int_\Omega W(\boldsymbol{r}'-\boldsymbol{r};h)\,dV' = 1. \tag{III.11}$$

As extensively discussed in [12], such a property is not satisfied when the kernel domain is not completely immersed inside the fluid domain which is a quite common configuration for those particles within a specific distance from the domain boundary, which in this paper will be a free surface(see figure 1). The incompleteness of the kernel close to the boundary will play a major role in this study.

The smoothing formula (III.10) can be applied to the gradient of a generic function

$$\langle\nabla f\rangle(\boldsymbol{r}) = \int_\Omega \nabla' f(\boldsymbol{r}')\,W(\boldsymbol{r}'-\boldsymbol{r})\,dV' \tag{III.12}$$

with ∇' meaning that the derivatives are computed on the $\boldsymbol{r}'$ variable. Equation III.12 can be further analyzed if it is

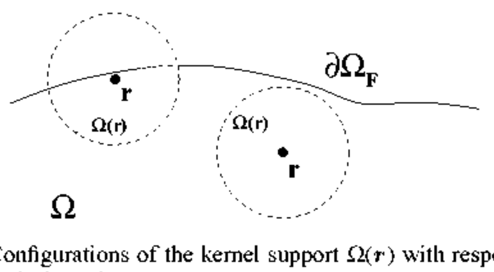

Fig. 1. Configurations of the kernel support $\Omega(\boldsymbol{r})$ with respect to the fluid domain boundary.

integrated by parts:

$$\langle \nabla f \rangle(\boldsymbol{r}) = \int_{\Omega} f(\boldsymbol{r}')\nabla W(\boldsymbol{r}'-\boldsymbol{r})\,dV' + \quad \text{(III.13)}$$

$$\int_{\partial\Omega} f(\boldsymbol{r}')\,W(\boldsymbol{r}'-\boldsymbol{r})\,\boldsymbol{n}'\,dS' \quad \text{(III.14)}$$

∇ indicates the derivatives with respect to the variable $\boldsymbol{r}$ and $\boldsymbol{n}'$ is a unitary normal vector of $\partial\Omega$ pointing outwards Ω. To obtain this equation, the antisymmetry property of the kernel gradient ($\nabla' W(\boldsymbol{r}-\boldsymbol{r}') = -\nabla W(\boldsymbol{r}-\boldsymbol{r}')$) has been used. Equation (III.12) implies that the gradient of any generic function can be obtained from the knowledge of the function itself using a smoothing procedure. If this procedure is applied to the differential operators of the continuum equations (II.1), shortening the notation $\langle f \rangle(\boldsymbol{r})$ by $\langle f \rangle$, the SPH continuous formulation of the Navier-Stokes equations in its weakly compressible form is obtained:

$$\begin{cases} \dfrac{D\rho}{Dt} + \rho\,\langle \nabla \cdot \boldsymbol{u} \rangle = 0\,, \\ \dfrac{D\boldsymbol{u}}{Dt} = \boldsymbol{g} - \dfrac{\langle \nabla p \rangle}{\rho} + \dfrac{\langle \nabla \cdot \mathbb{V} \rangle}{\rho}\,, \end{cases} \quad \text{(III.15)}$$

The consistency of (III.15) for the modeling of the Euler equations in the presence of solid boundaries and free surfaces has been investigated in Colagrossi et al [12]. Such analysis has been extended in this study for the convergence of the smoothed viscous term in Newtonian fluids.

IV. The SPH viscous term

The smoothed viscous term $\langle \nabla \cdot \mathbb{V} \rangle$ in equation III.15, which encompasses the computation of second order derivatives, can be modelled using different approaches. Without resorting to a direct second order differentiation, which has several disadvantages([4]), such computation is obtained by combining the formula for the first derivatives with a Taylor expansion of the velocity field (see Español & Revenga [7]). The core of this procedure is based on the assumption that the contributions associated with the first derivatives of the velocity field cancel out because of the symmetry properties of the kernel function and that only the contributions associated with the second derivatives remain. Unfortunately, this is true only inside the fluid domain where the kernel domain is complete. Actually, in the vicinity of the free surface, the kernel domain is truncated by the free surface and those symmetry properties are not valid any more. This implies that the first-order contributions of the Taylor expansions are not negligible and, therefore, that the viscous formulas might not converge close the free surface.

The procedure derived by Español & Revenga [7] is repeated in the following taking into account the presence of the free-surface This issue is completely general and would be enough to be prudent when using SPH expressions which approximate second derivatives. Furthermore, it will be shown in the present paper that the first-order contributions which appear near the free surface are generally divergent when the smoothing length goes to zero, that is, when we try to converge with SPH towards the continuum equations of fluid dynamics.

In the following section we show how to get the asymptotic expansion of the smoothed viscous term. The two most popular formulations have been considered: i) the Monaghan & Gingold [5] formula, ii) the Morris *et al.* formula [6]. Nonetheless, we focus on the Monaghan & Gingold formula because the procedure is general and the results for the Morris *et al.* formula [6] will be just enumerated without extensive demonstration.

A. Monaghan & Gingold formulation.

The continuous formulation of the Monaghan & Gingold viscous term is:

$$\begin{aligned} &< \nabla \cdot \mathbb{V} >^{MG}(\boldsymbol{r}) = \\ &\mu K \int_{\Omega} \frac{[\boldsymbol{u}(\boldsymbol{r}') - \boldsymbol{u}(\boldsymbol{r})]\cdot(\boldsymbol{r}'-\boldsymbol{r})}{|\boldsymbol{r}'-\boldsymbol{r}|^2}\,\nabla W(\boldsymbol{r}'-\boldsymbol{r})\,dV'\,, \end{aligned} \quad \text{(IV.16)}$$

$K = 6, 8, 10$, respectively in $1D$, $2D$ and $3D$.

1) Taylor expansion of the Monaghan & Gingold formulation.: In the following paragraphs the aim is to investigate the convergence properties of the smoothed viscous term $< \nabla \cdot \mathbb{V} >^{MG}$ in its continuous formulation. A significant part of the analysis which follows is based on the results of Español & Revenga [7]. In their work, using a Taylor expansion, they provided a general expression valid for the continuous formulation of the second order derivatives within the SPH framework. An analysis of that sort will be carried out for the smoothed viscous term together with a convergence assessment covering the different possibilities that could take place in regards to the position of a particle's centered kernel support $\Omega(\boldsymbol{r})$ and the free surface $\partial\Omega_F$. An asymptotic study depending on the value of h is the outcome of this analysis.

The spatial gradient of the kernel ∇W in equation IV.16 takes the compact form (IV.17) due to the kernel isotropy.

$$\begin{aligned} &\nabla W(\boldsymbol{r}'-\boldsymbol{r}) = -\frac{\boldsymbol{r}'-\boldsymbol{r}}{s}\,\frac{dW}{ds}\,, \\ &s = |\boldsymbol{r}'-\boldsymbol{r}| \end{aligned} \quad \text{(IV.17)}$$

Consistently with the used notation, $\boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{u}(\boldsymbol{r}')$ are respectively shortened as $\boldsymbol{u}$ and $\boldsymbol{u}'$. The origin of the frame of reference will be set as $\boldsymbol{r}$. After these considerations, the integral (IV.16) takes the following compact form

$$\begin{aligned} &< \nabla \cdot \mathbb{V} >^{MG} = \\ &-\mu K \int_{\Omega} \frac{(\boldsymbol{u}'-\boldsymbol{u})\cdot\boldsymbol{r}'}{|\boldsymbol{r}'|^3}\,\boldsymbol{r}'\,\frac{dW}{ds}\,dV'\,, \end{aligned} \quad \text{(IV.18)}$$

A Taylor expansion of the velocity field is performed

$$\boldsymbol{u}'-\boldsymbol{u} = \nabla\boldsymbol{u}\Big|_{\boldsymbol{r}}\cdot\boldsymbol{r}' + \tfrac{1}{2}\,\boldsymbol{r}'\cdot\mathbb{H}\Big|_{\boldsymbol{r}}\cdot\boldsymbol{r}' + O(|\boldsymbol{r}'|^3)\,, \quad \text{(IV.19)}$$

in which $\mathbb{H}|_{\boldsymbol{r}}$ denotes the third-order Hessian tensor. Consistently again with the notation, $\nabla\boldsymbol{u}\Big|_{\boldsymbol{r}}$ is shortened as $\nabla\boldsymbol{u}$

and $\mathbb{H}\Big|_{r}$ as $\mathbb{H}$. After using this notation and rearranging some terms, equation (IV.18) becomes

$$< \nabla \cdot \mathbb{V} >_i^{MG} = \mu \Big[\mathbf{F}_{ijk} \mathbb{D}_{jk} + \mathbb{G}_{ijkl} \mathbb{H}_{jkl} \Big] + O(h) \qquad \text{(IV.20)}$$

where the tensor $\mathbf{F}$ and $\mathbb{G}$ are defined as:

$$\mathbf{F} = -K \int_{\Omega} \frac{\boldsymbol{r}' \otimes \boldsymbol{r}' \otimes \boldsymbol{r}'}{|\boldsymbol{r}'|^3} \frac{dW}{ds} dV' ,$$

$$\mathbb{G} = -\frac{K}{2} \int_{\Omega} \frac{\boldsymbol{r}' \otimes \boldsymbol{r}' \otimes \boldsymbol{r}' \otimes \boldsymbol{r}'}{|\boldsymbol{r}'|^3} \frac{dW}{ds} dV' . \qquad \text{(IV.21)}$$

Using the properties of the tensors $\mathbb{F}$ and $\mathbb{G}$, the convergence of the smoothed viscous term can be now assessed, as previously announced, in what respect to the relative position of the particles and the fluid domain boundary.

2) Convergence of the Monaghan & Gingold formulation inside the fluid domain: For each point inside the fluid domain Ω, it is possible to find a specific value of the smoothing length such that the kernel support centered in that point will be completely immersed in Ω, without any intersection with the domain boundary. In this case the tensor $\mathbb{F}$ is equal to zero and the smoothed viscous term (IV.20) consequently becomes:

$$< \nabla \cdot \mathbb{V} >_i^{MG} = \mu \mathbb{G}_{ijkl} \mathbb{H}_{jkl} + O(h)$$

$$= \mu \left[\mathbb{J}_{ijkl} \mathbb{H}_{jkl} + \frac{\partial^2 (\boldsymbol{r}_i \mathbb{H}_{jkl} \boldsymbol{r}_j)}{\partial \boldsymbol{r}_k \partial \boldsymbol{r}_l} \right] + O(h) \qquad \text{(IV.22)}$$

where $\mathbb{J}_{ijkl} = \delta_{ij} \delta_{kl}$ (see for example [13]) and δ denotes the Kroneker tensor. Since

$$\left[\nabla^2 \boldsymbol{u} \Big|_r \right]_i = \mathbb{J}_{ijkl} \mathbb{H}_{jkl} \qquad \text{and}$$

$$\left[\nabla (\nabla \cdot \boldsymbol{u}) \Big|_r \right]_i = \frac{1}{2} \frac{\partial^2 (\boldsymbol{r}_i \mathbb{H}_{jkl} \boldsymbol{r}_j)}{\partial \boldsymbol{r}_k \partial \boldsymbol{r}_l} \qquad \text{(IV.23)}$$

it follows that

$$\lim_{h \to 0} < \nabla \cdot \mathbb{V} >^{MG} = \mu \nabla^2 \boldsymbol{u}(\boldsymbol{r}) + 2 \mu \nabla (\nabla \cdot \boldsymbol{u})(\boldsymbol{r}) \qquad \text{(IV.24)}$$

In order to have this expression, that had been previously reported by Español & Revenga [7], consistent with the continuous viscous stresses definition (II.5), the Lamé constants λ and μ must have the same value in the continuous smoothed representation of the fluid (IV.16). This also means that the formula (IV.16) does not satisfy the Stokes hypothesis ($\lambda = -2\mu/3$). The consequences of this fact, have not received much attention in the literature and will be left for future studies.

3) Convergence of of the Monaghan & Gingold formulation at the free surface: If the material point belongs to the free surface (that is, $\boldsymbol{r} \in \partial\Omega_F$), the value of the components of the tensor $\mathbb{G}$ are reduced due to the truncation of the kernel support. In particular if the free surface in the vicinity of the point $\boldsymbol{r}$ is a regular manifold, it will be possible to approximate it by a flat surface through its tangent plane. This way, by taking the limit as the smoothing length h goes to zero, it can be shown that all the components of the tensor $\mathbb{G}$ will halve their values.

Apart from halving the value of the components of the tensor $\mathbb{G}$, the most relevant issue in this configuration is that the tensor $\mathbb{F}$ is not null but it becomes singular, diverging like h^{-1}. It is possible to theoretically prove that if the fluid is incompressible such singularity will manifest in those flows in which the component of the stress $\boldsymbol{t}$ (eq. II.7) that is perpendicular to the free surface is not identically zero (*i.e.* $\boldsymbol{n} \cdot \partial \boldsymbol{u} / \partial \boldsymbol{n} \neq 0$). For sake of brevity in this work we show this behavior only through numerical examples leaving the mathematical demonstration for future works.

It is remarkable that the singularity of the smoothed viscous term is not a consequence of the weakly compressibility assumption but is caused by the truncation of the kernel domain at the free surface. This makes this discussion relevant even for those interested in pure incompressible approaches [14].

B. Morris et al. *formulation.*

The second expression of the viscous term is the Morris *et al.* formula which in its continuous form reads:

$$< \nabla \cdot \mathbb{V} >^{Mo} (\boldsymbol{r}) =$$

$$2\mu \int_{\Omega} \frac{(\boldsymbol{r}' - \boldsymbol{r}) \cdot \nabla W(\boldsymbol{r}' - \boldsymbol{r})}{|\boldsymbol{r}' - \boldsymbol{r}|^2} \left[\boldsymbol{u}(\boldsymbol{r}') - \boldsymbol{u}(\boldsymbol{r}) \right] dV' . \qquad \text{(IV.25)}$$

Using the Taylor expansion described in (IV.19), we get:

$$< \nabla \cdot \mathbb{V} >_i^{Mo} = \mu \left[\nabla \boldsymbol{u}_{ij} \mathbb{N}_j + \mathbb{H}_{ijk} \mathbb{M}_{jk} \right] + O(h) , \qquad \text{(IV.26)}$$

where:

$$\mathbb{N}_j = -2 \int_{\Omega} \frac{\boldsymbol{r}'_j}{s} \frac{dW}{ds} dV' , \quad \mathbb{M}_{jk} = - \int_{\Omega} \frac{\boldsymbol{r}'_j \boldsymbol{r}'_k}{s} \frac{dW}{ds} dV' . \qquad \text{(IV.27)}$$

Similarly to the Monaghan & Gingold formula, $\mathbb{N}$ is zero inside the fluid domain while diverges with h^{-1} near the free surface Regarding the second-order terms, $\mathbb{M}_{jk} = \delta_{jk}$ inside the fluid while it halves near the free surface. This conclusion does not depend on the kernel shape.

Inside the fluid domain the Morris *et al.* formula gives:

$$\lim_{h \to 0} < \nabla \cdot \mathbb{V} >^{Mo} = \mu \nabla^2 \boldsymbol{u} . \qquad \text{(IV.28)}$$

Note that the Morris *et al.* formula approximates the exact viscous term for incompressible flows while the Monaghan & Gingold one also encompasses extra terms related with the divergence of the velocity field which should be zero for incompressible flows.

V. Dissipative effects

In order to assess the influence in global dynamics of the singularities documented in section IV, the global dissipative effects associated to the viscous formulations by Monaghan & Gingold and Morris *et al.* will be now considered, in the presence of a free surface. For the sake of simplicity, all the results shown in the following are valid under the assumption that the fluid is incompressible.

The dissipative term is obtained from the viscous terms by evaluating the integral [15]:

$$\int_\Omega \boldsymbol{u} \cdot < \nabla \cdot \mathbb{V} > dV . \tag{V.29}$$

From a theoretical point of view, integral V.29 can be integrated by parts obtaining a boundary term, associated to the power of the surface forces, and a bulk term which, from the second principle of Thermodynamics, is never negative and, therefore, causes the loss of kinetic energy of the fluid body. This is the dissipative term.

Since, by definition, along a free surface the tension is zero, the power of the boundary forces is expected to be null and, therefore, equation (V.29) allows us to get the whole dissipative term. From a mathematical point of view, the assumption made above induces some proper symmetry properties which lead to the following results:

$$\int_\Omega \boldsymbol{u} \cdot < \nabla \cdot \mathbb{V} >^{MG} dV = -2\mu \int_\Omega \mathbb{D}:\mathbb{D}\, dV + O(h) , \tag{V.30}$$

$$\int_\Omega \boldsymbol{u} \cdot < \nabla \cdot \mathbb{V} >^{Mo} dV = -\mu \int_\Omega \|\nabla \boldsymbol{u}\|^2 dV + O(h) . \tag{V.31}$$

The Monaghan & Gingold term provides an accurate result but the dissipative term associated to the Morris *et al.* formula is different from zero even in the case of a pure rigid rotation (that is, $\mathbb{D} \equiv 0$) where no dissipation should take place. This aspect will be further inspected in section VI where proper test cases are defined to show the behavior of the different viscous terms.

The results shown in the present section and in the previous one underline that the influence of the free surface and, consequently, of the singular terms on both the fluid kinematics and dynamics is difficult to analyze. Near the free surface, both the Monaghan & Gingold and the Morris *et al.* formulas show to be singular as the spatial resolution increases. This is expected to locally influence the kinematic and the evolution of the free surface. Surprisingly, with respect to the global fluid dynamics, the Monaghan & Gingold formulation gives the correct results in terms of dissipation while the Morris *et al.* one fails when considering free-surface flows. In the following section, some test cases to be simulated with an SPH code are considered. They may help at visualizing the results of the continuous analysis of this and previous sections.

VI. Test cases

In the present section we aim at discussing to what extent the features of the continuous smoothed viscous term are valid at the discrete level. This has been done by selecting two test cases for free-surface flows with distinct nature in regards to the onset of the singular behavior described in section IV In all the numerical simulations shown in the following we adopted a renormalized Gaussian kernel (see for example [16]). The support of the kernel is equal to $3h$ and the initial distance between the particle is indicated with dx. The ratio h/dx has been kept constant and equal to $4/3$. For the SPH simulations, the two formulations discussed in the paper have been used, that is: (i) the Monaghan & Gingold one $< \nabla \cdot \mathbb{V} >^{MG}$ (see equation IV.16) (ii) and the Morris *et al.* formula $< \nabla \cdot \mathbb{V} >^{Mo}$ (see equation IV.25). The results obtained through the different viscous models have been compared with analytical or reference solutions.

A. Evolution of a circular patch of fluid with a non-uniform initial vorticity distribution

The interest of this case arises, as mentioned, from its relevance in regards to the onset of the singular behavior of the viscous term, as discussed in section IV-A3. It is shown that the condition necessary for the onset is not fulfilled and the SPH simulation behaves accordingly.

A fluid circular cylinder of radius $R = 1$ subjected to a radial force field is considered. The origin of the frame of reference is the cylinder center and the radial force is $\boldsymbol{g} = -\beta^2 r\, \hat{\boldsymbol{r}}$ where β is a constant parameter, r is the radial coordinate and $\hat{\boldsymbol{r}}$ is the radial unit vector. The angular coordinate is denoted through θ while $\hat{\boldsymbol{\theta}}$ indicates the tangential unit vector. The sub-indexes r and θ are used to denote respectively the radial and angular components of the velocity field.

An incompressible isotropic (null θ derivatives) solution is sought. Under these hypotheses the Navier-Stokes equations reduce to:

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} - \beta^2 r , \tag{VI.32}$$

$$\frac{\partial u_\theta}{\partial t} = \nu \left\{ \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} \right\} . \tag{VI.33}$$

Since the angular velocity $\omega(r,t) = u_\theta(r,t)/r$, equation (VI.33) can be cast in the following format:

$$\frac{\partial \omega}{\partial t} = \frac{\nu}{r}\left\{ 3\frac{\partial \omega}{\partial r} + r\frac{\partial^2 \omega}{\partial r^2} \right\} . \tag{VI.34}$$

The initial distribution for the angular velocity that does not depend on θ, accordingly with the initial hypothesis, has been defined as:

$$\omega(r,0) = \omega_0 \left\{ \frac{\ell^2}{\ell^2 + r^2} + 2\frac{\ell^2 R}{(\ell^2 + R^2)^2}\, r \right\} \tag{VI.35}$$

The values chosen for the parameters follow: $\ell^2 = 0.1\,\mathrm{m}^2$, $\omega_0 = 1\,\mathrm{rad/s}$ and $\beta = \pi/8\,\mathrm{s}^{-1}$. The kinematic viscosity ν is equal to $10^{-3}\ m^2/s$ and the Reynolds number is $Re = R\, u_\theta(R,0)/\nu = 256$. With these values, the motion keeps stable, i.e., no dependence on θ develops during the flow evolution.

For the problem at hand, the condition (II.9) is equivalent to $\partial\omega/\partial r = 0$ at the boundary. It can be easily checked that this assumption is satisfied by the initial condition (VI.35). Note that the equation (VI.34) is not coupled with the pressure equation and, therefore, can be solved numerically to get a

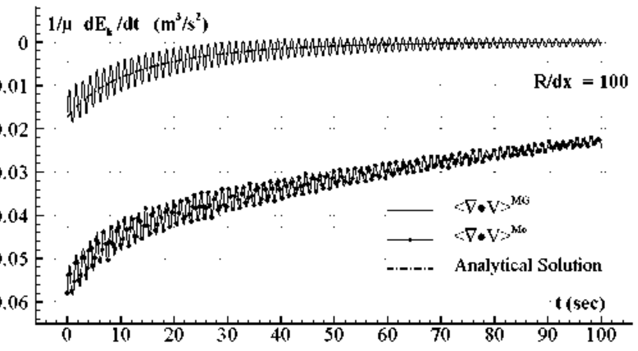

Fig. 4. Evolution of a circular patch of fluid with a non-uniform initial vorticity distribution. Time histories of the kinetic energy dissipation dE_k/dt evaluated using $< \nabla \cdot \mathbb{V} >^{MG}$, $< \nabla \cdot \mathbb{V} >^{Mo}$. The dashed-dotted line represents the analytical solution.

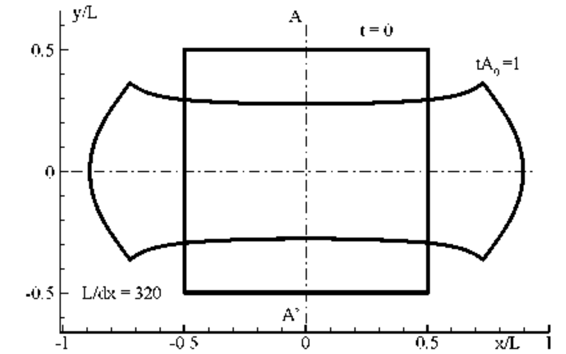

Fig. 5. Stretching of a squared patch of viscous fluid. Left: free surface configuration at time $t = 0$ and $tA_0 = 1$.

tends to stretch the domain Ω along the x-axis and to contract it in the y-direction (figure 5). The initial velocity field satisfies the boundary condition $\boldsymbol{\tau} \cdot \mathbb{D}_0 \cdot \boldsymbol{n}$. The initial pressure field is evaluated solving the Poisson equation for incompressible flow $\nabla^2 p = -2\rho A_0^2$ (see *e.g.* [17]). The pressure boundary condition along the free surface is $p = -2\mu\, \boldsymbol{n} \cdot \mathbb{D}_0 \cdot \boldsymbol{n}$ and the Reynolds number is $Re = LA_0/\nu = 20$. Due to this low value, about the 80% of the initial kinetic energy is dissipated in the interval $tA_0 = [0, 1]$. Since $\boldsymbol{n} \cdot \mathbb{D} \cdot \boldsymbol{n}$ is not null we expect that the viscous smoothing operators $< \nabla \cdot \mathbb{V} >$ to be singular at the free surface. This can be clearly observed in the top panel of figure 5 where the y-component of the vector $< \nabla \cdot \mathbb{V} >^{MG}$ along the vertical symmetric axis has been evaluated decreasing the smoothing length h. The operator $< \nabla \cdot \mathbb{V} >^{Mo}$ show a similar behavior (figure6 bottom).

Contrary to the first test case, it is not feasible to obtain an analytic solution for the problem at hand. A reference solution to be used as comparison with the SPH solver has been obtained numerically using a finite difference method with a deformable mesh (hereinafter simply indicated as FDM). Such a reference solution has been obtained using the highest SPH spatial resolution $L/dx = 320$ for the mesh size.

Top panel of figure 7 shows the time history for the kinetic

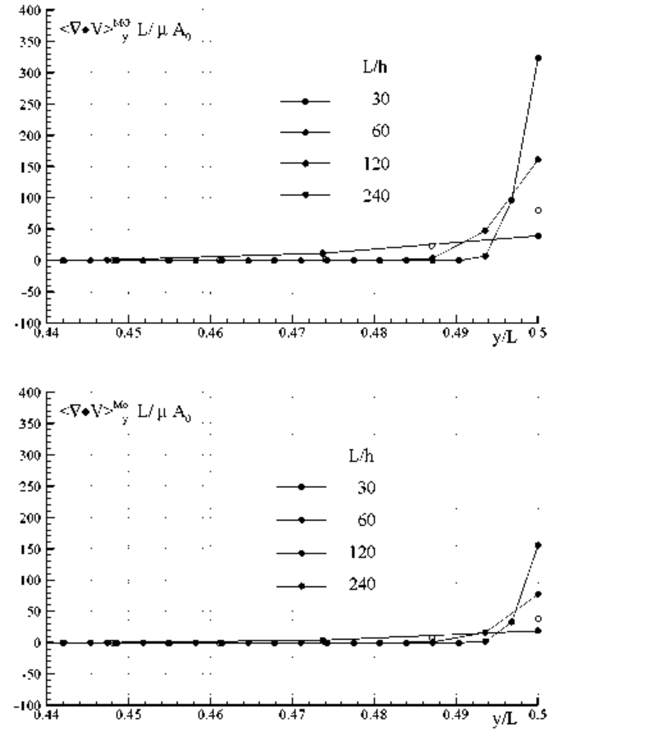

Fig. 6. Stretching of a squared patch of viscous fluid. Top: enlarged view of the y-component of the vector $< \nabla \cdot \mathbb{V} >^{MG}$ (see equation IV.16) evaluated along the symmetry axis AA' at the initial time $t = 0$ for decreasing smoothing length h . Bottom: like the left panel using the operator $< \nabla \cdot \mathbb{V} >^{Mo}$.

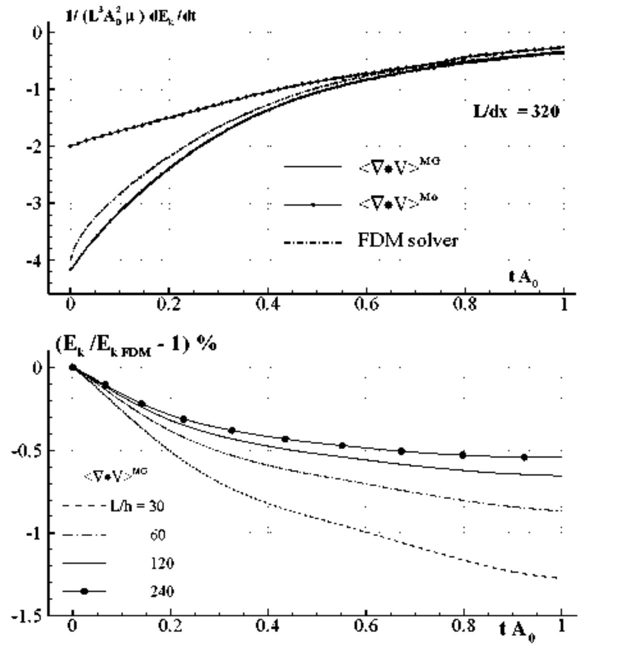

Fig. 7. Stretching of a squared patch of viscous fluid. Top: time histories of the kinetic energy dissipation dE_k/dt using formulae IV.16 and IV.25. The dashed-dotted line represents the solution obtained through a Finite Difference Scheme solver. Bottom: relative error between the SPH kinematic energy evaluated through IV.16 and that obtained by the FDM solver.

energy dissipation rate. The operator $<\nabla\cdot\mathbb{V}>^{MG}$ provides fair agreement between the dissipation predicted by the FDM and that obtained through the SPH solver. Conversely, using the Morris *et al* formulation, the viscous dissipation is largely underestimated. The exact kinetic energy dissipation at initial time $t=0$ is equal to $dE_k/dt = -4\mu A_0^2 L^3$. The error made using $<\nabla\cdot\mathbb{V}>^{MG}$ is about 5%. With the operator $<\nabla\cdot\mathbb{V}>^{Mo}$ the initial value of dE_k/dt is halved as theoretically predicted by the equation V.31. The bottom panel of figure 7 shows the relative error on the kinetic energy between the FDM solver and the SPH one. In this figure the operator $<\nabla\cdot\mathbb{V}>^{MG}$ is computed using different smoothing length values (*i.e.* for different spatial resolutions since h/dx is fixed and equal to $4/3$). The error is less than 1% for the three higher resolutions and reduce to 0.55% for the highest one.

VII. Conclusions

A theoretical analysis of the two most popular SPH formulations for Newtonian viscous terms, i.e., Monaghan & Gingold [5] and Morris *et al.* [6], when used to simulate free surface flows, has been carried out in this paper. It has been demonstrated that when the normal, to the free surface, component of the stress is not null, the viscous term becomes singular with the inverse of the smoothing length. In order to assess the influence of such singular behavior in the global dynamics of the flow, the energy dissipation due to viscosity has been studied. It has been shown that the free-surface singularities vanish when the integral quantities are considered and give a bounded value of the mechanical energy dissipation rate. Notwithstanding that, the Morris *et al.* [6] formulation fails in the correct prediction of the dissipative effects when free-surface viscous flows are considered. This implies that such a formulation may not be adequate to describe free-surface viscous flows.

Finally two test cases aimed at demonstrating the validity of the conclusions obtained from the continuous analysis have been presented. The results from these two cases, a rotating patch of viscous fluid and the stretching of a square patch of fluid, agree with the results found in the theoretical parts of the paper.

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